**The Time Value of Money – Chapter 4 in RWJJ**

The time value of money - The most basic concept in Finance.

**Definitions:**

**Interest** - Money paid for the use of your money. Expressed as a % or a decimal.

**Future Value** - Amount to which an investment will grow after earning interest.

**Present Value** – Amount of money you start with – the initial investment.

**Simple Interest** - Interest earned on the original investment

**Compound Interest** - Interest earned on interest

**Example**: Interest = Interest Rate x Present Value (your initial investment)

 $6 = .06 · $100

Value after one year = $100 + $6 = $106

Let r = interest rate; PV = Present Value and FV = Future Value

Value after one year = PV (1+r)

 $100 (1 + .06) = $106

Second Year - Start with $106

 Interest = $106 ·.06 = $6.36

Value at end of year = $106 + $6.36 = $112.36

 = $106 (1 + r)

 = $106 (1 + .06) = $112.36

Start at beginning to year one and go to the value at the end of year 2.

100\_\_\_\_\_\_\_\_\_\_\_106\_\_\_\_\_\_\_\_\_\_\_\_\_112.36

 0 1 2

Value after 2 years = $100 (1.06) (1.06) = $112.36

 = $100 (1.06)2 = $112.36

 = PV (1+r) (1+r)

 = PV (1+r)2

FV = PV (1+r)t

where t = number of compounding periods

and r = interest rate per compounding period

Taking it out to 5 years:

100\_\_\_\_\_\_106\_\_\_\_\_\_\_112.36\_\_\_\_\_\_\_119.10\_\_\_\_\_\_\_\_126.25\_\_\_\_\_\_\_133.82

 0 1 2 3 4 5

FV = PV (1 + r)t Here r = .06 = 6%

for t = 3 FV = 100 (1.06)3 = $119.10

for t = 4 FV = 100 (1.06)4 = $126.25

for t = 5 FV = 100 (1.06)5 = $133.82

On **Excel**:

Locate Future Value with the Function Wizard

1. Enter .06 or 6% as Rate
2. Enter 5 as Nper
3. Leave Pmt blank
4. Enter 100 as PV
5. Leave Type blank
6. Note that if you enter PV as a positive value, FV will be negative

FV = PV [(1+r)t] where [(1+r)t] is the future value of $1.00. This is sometimes called the Future Value Interest Factor (FVIF)

**Another example**: Invest $25 for 2 years at 9%. What will it grow to?

FV = PV (1+r)t

 = $25 (1.09)2

 = $25 (1.09) (1.09) = $29.70

**Excel**: FV function

Enter .09 or 9% as Rate

Enter 2 as Nper

Leave Pmt blank

Enter 25 as PV

Leave Type blank

If you want to see FV as a positive value, either enter -25 for PV or solve for -FV

 25 (1.09) 25 (1.09)2

25 27.25 29.70

 0 1 2

Try $10 at 5% for 30 years

FV = PV (1+r)t

 = 10 (1.05)30

 = 10 (4.3219) = $43.22

Note that t can be any time period (month, week, year, quarter, etc.) – it is the number of compounding periods and r is the interest rate **per compounding period**.

**Compounding Period**: How often interest is posted. Immediately after it is posted, you start earning (or paying) interest on the interest.

**Example**: Credit cards: Interest accrues monthly

 Monthly rate = 1.5%

 You charge $100 on your card

 You wait 2 years to pay it off - what do you owe?

FV = PV (1+r)t where r = 1.5% (interest rate per month) and t = 24 months

 = 100 (1.015)24

 = $142.95 = $100 principal plus $42.95 interest

Question: If 1.5% = monthly rate, what is the annual rate?

**APR =** **Annual Percentage Rate:** the most commonly used way to express interest rates.

APR = r · m where r = int. rate per compounding period and m = number of compounding periods in a year.

1.5% / month = 18% APR because (1.5) (12) = 18.

 However, this ignores the compounding that takes place during the year and doesn’t give you the same result at the end of two years that we got earlier.

If we use an annual rate of 18%: FV = 100 (1.18)2

 = $139.24 which is not $142.95

This is a general formula where you are compounding m times per year for n years. Since APR = (r) (m), it must be that APR/m = r. So the above formula is equivalent to

FV = PV (1+r)t

**EAR = Effective Annual Rate**: The annual growth rate of money allowing for all the effects of compounding. This is sometimes also called the **APY** (Annual Percentage Yield)

In our credit card example, EAR is the annual rate which will give us the correct future value of $142.95 with the formula FV = PV (1+r)t when t = 2 years.

(1 + EAR) = (1 + monthly rate)12 when we have monthly compounding

(1 + EAR) = (1 + daily rate)365 when we have daily compounding

(1 + EAR) = (1 + semiannual rate)2 when we have semiannual compounding

where m is once again the number of compounding periods in a year.

**APR** is the annual growth rate of funds using simple interest.

 Calculate it as 1.5% / mo. · 12 mo. = 18% / yr. (APR)

**EAR** includes all compounding within the year

(1+EAR) = (1+monthly rate)12

(1+EAR) = (1 + .015)12

(1.015)12 = 1.19562

EAR = 19.562%

Note that 100 (1.19562)2 = 142.95

so: r = 1.5% and t = 24 mo.

This is the same as: r = 19.562% and t = 2 yrs.

But not the same as: r = 18% and t = 2 yrs.

APR is the most commonly used interest rate. Whenever you see an interest rate, you should assume it is the APR unless it is specified otherwise.

APR/m gives you the interest rate per compounding period because APR is found by multiplying the interest rate for the compounding period by m.

**Converting APR to EAR**

Compounding Period 1 + EAR

Annual 1+APR

Semi-Annual (1 + APR/2)2

Quarterly (1 + APR/4)4

Monthly (1 + APR/12)12

Weekly (1 + APR/52)52

Daily (1 + APR/365)365

Continuously er where r is the stated annual rate (APR)

**10% APR Compounded over different time periods. What is the EAR?**

 (per./yr) (10/t) (.10/t)

Comp. PER t % r (1+r)t EAR

Annual 1 10 .10 1.1 = 1.10 10%

Semi-Annual 2 5 .05 (1.05)2 = 1.1025 10.25

Quarterly 4 2.5 .025 (1.025)4 = 1.1038 10.38

Monthly 12 .83 .0083 (1.0083)12 = 1.1047 10.47

Weekly 52 .192 .00192 (1.00192)52 = 1.1051 10.51

Daily 365 .0274 .000274 (1.000274)365 = 1.105165 10.5165

Continuously ∞ e.1 1.10517 10.517

Note what happens when you take this to the limit:  as t → ∞, the value approaches 2.718……. the value of e.

**Continuous compounding** - use e = 2.718281828... = approx. 2.718

FV = PV(ert)

Note that if r = 10% and t = 1; e.1 = (2.718).1 = 1.10517 ⇒ 10.517% = EAR

On Excel, use the “exp” function with r·t as the argument to find the FV

To find the continuously compounded rate of return for a given time period, use 



Example: Your investment is worth $100 on March 1 and it is worth $102 on April 1.

The monthly return is 2% if there was monthly compounding. This means that it grew from $100 to $102 at the end of the month (on April 1).

The continuously compounded monthly return was  = .0198 = 1.98%

Note that this assumes your investment was continually growing (from $100 to $102) over the course of the month. It did not remain at $100 till the end of the month and then jump to $102.

How much money will you have at the end of the year?

With monthly compounding, FV = PV (1+r)t = 100 (1.02)12 = 126.82

With continuous compounding, FV = PV(ert) = 100(e(.0198)(12)) = 126.82

What is the FV of $100 in 3 yrs. with 10% APR (stated rate) with the following compoundings:

Annually: 100(1.1)3 = $133.10

Semi-Annually: 100(1.05)6 = $134.01

Monthly: 100(1.0083)36 = $134.82

Continuously: 100[(e)(.1)(3)] = $134.99

Note they all have 10% APR (stated rate) but different EARs

Example: 8% APR with monthly compounding.

 What is the effective annual rate?

 (1 + .08/12)12 = 1.083. So, EAR = 8.3%

We can take the equation: FV = PV (1+r)t and rearrange it to get

PV = FV\_\_\_

 (1+r)t

PV = value today of a future cash flow

r = discount rate (interest rate)

To calculate PV we discount FV at interest rate r over t periods

**Example**: How much to invest today for it to grow to $500 in 2 years if the interest rate is 7%?

PV = \_FV\_\_

 (1+r)t

 = 500 .

 (1.07)2

 = 436.72

Invest $436.72 today at 7% to get $500 in 2 years.

436.72\_\_\_\_\_\_\_\_\_467.29\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_500.00\_\_

 0 1 2

**Another Example**:

What is the value today of $50,000 to be received in 10 years if the interest rate is 9% (APR) and we have monthly compounding?

Excel: Find PV under function wizard

Rate = .09/12

Nper = 10 x 12

Pmt = blank

Fv = 50,000

Type = blank

PV = \_FV\_

 (1+r)t

 = 50,000

 

 = 20,396.87

So far, we have 2 basic equations:

1. FV = PV (1+r)t

2. PV = \_\_FV\_\_

 (1+r)t

How can we solve for “r”?

Take (1) and divide each side by PV and then reverse sides to get (1+r)t = FV

 PV

Now raise each side to the power of 1/t to get 1+r =  This is how we find r.

r =  - 1

Suppose someone tells you if you give him $100 today, he’ll give you $120 in 3 years. Is this a good deal or a bad deal? We must determine what interest rate we are getting.

But note: To solve this, we must make an assumption about how frequently our money is being compounded.

PV = $100 FV = $120 t = 3 years ( if we assume annual compounding)

100 120

 0 1 2 3

r =  - 1

 = - 1

 = (1.2)1/3 - 1

 = .0627 = 6.27%

Using Excel: Find Rate under the function wizard

Nper = 3

Pmt = blank

Pv = -100

Fv = 120

Type = blank

**Solving for t:**

FV = PV (1+r)t

FV = (1+r)t

# PV

ln  = [ln (1+r)] ⋅ t

ln  = t

 ln (1+r)

**Example**:

How long will it take a $1,000 investment to grow to $100,000 if it earns 14% per year with annual compounding?

ln  = 4.60517 / 0.131 = 35.146 years

 ln (1.14) 1,000 100,000

 0 ?

Excel: Find NPER on function wizard

Rate = .14

Pmt = blank

Pv = -1000

Fv = 100000

Type = blank

Note though that if interest is posted at the end of the year, you won’t have it until the end of year 36!

**Summing the Present Values**

You are the agent for a professional athlete. Two contract alternatives are presented to you. Which is better?

These are the cash flows your client is guaranteed to receive:

Contract A: $1 million per year for five years with each payment coming at the end of the year.

Contract B: $3.1 million at the end of the first year and $400,000 at the end of each of the following four years.

Under contract A, your athlete will receive a total of $5 million

Under contract B, your athlete will receive a total of $4.7 million

Unfortunately, you can’t add money received in different time periods - even if they are paired up in the same years like this.

You must get all the money valued at the same time period. We’ll use time zero (today).

If r = 10%

1. $1 mill. = $909,091 B) $3.1 mill = $2,818,182

 (1.1)1 (1.1)1

 $1 mill = $826,446 $400,000 = $330,579

 (1.1)2 (1.1)2

 $1 mill = $751,315 $400,000 = $300,526

 (1.1)3 (1.1)3

 $1 mill = $683,013 $400,000 = $273,205

 (1.1)4 (1.1)4

 $1 mill = $620,921 $400,000 = $248,369

 (1.1)5 (1.1)5 .

 $3,790,786 $3,970,860

So if r = 10%, PVB > PVA

In Excel, you can solve for NPV (*Net* Present Value) to get these answers

**Excel**: Find NPV on function wizard

Rate = .10

Values must be entered as: 1,000,000, 1,000,000, 1,000,000, 1,000,000, 1,000,000

or as 3,100,000, 400,000, 400,000, 400,000, 400,000

If r = 2%

A) $1 mill. = $980,392 B) $3.1 mill = $3,039,216

 (1.02)1 (1.02)1

 $1 mill = $961,169 $400,000 = $384,468

 (1.02)2 (1.02)2

 $1 mill = $942,322 $400,000 = $376,929

 (1.02)3 (1.02)3

 $1 mill = $923,845 $400,000 = $369,538

 (1.02)4 (1.02)4

 $1 mill = $905,731 $400,000 = $362,292

 (1.02)5 (1.02)5 .

 $4,713,460 $4,532,443

So if r = 2% then PVA > PVB

Notice that the higher the interest rate, the more important it is to get money early on.

The main point is that if you wish to compare money received in different time periods, you must first convert all CF to PV.

You can always add PVs but you can never add money in different time periods unless the interest rate is zero.

Interest Rate is 5%. What is the total PV of: $1,000 at end of year 1,2,3,4 &5?

PV = $1,000 + $1,000 + $1,000 + $1,000 + $1,000

 1.05 1.052 1.053 1.054 1.055

 = 952.38 + 907.03 + 863.84 + 822.70 + 783.53

 = $4,329.48

Excel: Find PV on function wizard

Rate = .05

Nper = 5

Pmt = 1,000

Fv = blank

Type = blank

**Here’s another way to look at it:**

$4,329.48 will provide $1,000/year for 5 years if the interest rate is 5%.

Year Start Interest Earned Total Payout Remainder

1 $4,329.48 $216.47 $4545.95 $1000 $3545.95

2 $3,545.95 177.30 3723.25 1000 2723.25

3 $2,723.25 136.16 2859.41 1000 1859.41

4 $1,859.41 92.97 1952.38 1000 952.38

5 $ 952.38 47.62 1000.00 1000 0

So, at 5%, how much do we need to provide $1,000/year **forever**?

If it’s to go forever, we can’t use up any principal, so, we must produce $1,000 in interest each year.

# At 5%, how much principal is needed to produce $1,000 in interest?

Interest = Interest Rate x Principal

Principal = \_\_\_Interest\_\_\_ = $1,000 = $20,000

 Interest Rate .05

So $20,000 is the principal needed to produce $1,000 interest at 5%

$20,000 · 5% = $1,000 - Pay the $1,000 and start over each year.

**Perpetuity** - A stream of level cash flows that never ends.

PV of Perpetuity = Cash Flow or, PVP = C

 Int. Rate r

**Annuity** - Stream of level cash payments that ends. It’s not an annuity if the payments are not all equal.

$1 million lottery is an annuity of $50,000/year for 20 years.

Earlier example was an annuity: $1,000/year for 5 years. Remember what the PV was?

PV = $1,000 + $1,000 + $1,000 + $1,000 + $1,000 = $4,329.48

 1.05 1.052 1.053 1.054 1.055

There is a quicker way:

 Year: 1 2 3 4 5 6 7 8 ...........

Perpetuity A $1 $1 $1 $1 $1 $1 $1 $1 ...........

Perpetuity B $1 $1 $1 ...........

A 5 year annuity = Perpetuity A - Perpetuity B

Perpetuity A = 1 Perpetuity B = 1 · \_\_\_1\_\_\_ = value at the

 r r (1+r)5 beginning of yr. 5

 discounted to today

PVA =  The expression in brackets is called the annuity factor.

From our previous example:PVA = 

 = $4,329.48 which is the same result we got previously

Note that in an ordinary annuity, the present value is the value at one compounding period **before** the first cash flow. For example – if the first cash flow is at time period 1, the PV of an ordinary annuity will give you the value at time zero.

Example: State lottery: How much is $50,000/year for 20 years really worth?

 Assume a 7% interest rate.

PVA =  = 

Excel: Nper = 20

 Rate = 7

 PMT = -50,000

 Solve for PV = 529,700.71

But, in the lottery, you get your first payment immediately and then the 19 others. In other words, you get each payment at the **beginning** of the year, rather than (as we usually do) at the end of the year. This is called an **Annuity Due**.

With your calculator, just put it in the “Begin” mode.

With Excel, enter “1” for type.

PVAD = $50,000 + 

 = 50,000 + 516,779.76 = 566,779.76

Also: Annuity Due = Ordinary Annuity (1+r)

 $529,700.71 (1.07) = $566,779.76

PVAD = C  or  (1+r)

Note that it’s higher than when you get payments at the end of each year.

If you win the Lottery, should you take $500,000 up front, or take annuity payments of $50,000 per year?

You must first determine what is the implicit interest rate in the annuity.

If you take the $500,000 now, PV = 500,000

At what value of ‘r’ is the lottery annuity PV equal to 500,000?

500,000 = 50,000 + 50,000

450,000 = 50,000

Solve for r with Excel and you get 8.92%

So the Lottery Annuity has an implicit interest rate of 8.92%

Can you get 8.92% (virtually risk-free) for 19 years if you take the lump sum?

If not, take the annuity.

**Home Mortgages**: **Example**:

Amount of the Mortgage = $200,000

Obtain a 30 year mortgage at 6% APR

Monthly Payments

PVA = 

$200,000 =   = 

C = $1,199.10 = monthly mortgage payment

In Excel, you are solving for the payment.

End of first month: Original Principal = $200,000

 Interest due = 6%/12 = .5% = .005

 .005 · $200,000 = $1,000

 You pay $1,199.10

 Interest 1,000.00

 Princ. $ 199.10

End of second month: New Princ. = $200,000 - $199.10 = $199,800.90

 Int. Due = .005 x $199,800.90 = $999.00

 You pay $1,199.10

 Interest 999.00

 Princ. $ 200.10

For 30 years, each month, the interest decreases and the principle increases. On the last month, you pay off all the principle with only a few dollars of interest.

Interest is due on the unpaid balance each month - not on the entire $200,000.

This is called **amortizing a loan**. Some money is paying the principal and some is paying the interest.

Anytime we want to know what the payoff of our loan is, we merely have to look at the present value of the remaining future payments.

How much do we still owe on the loan after 10 years of payments (20 years remaining)?

PVA = 

 = $167,371.45

**A Delayed Annuity**

What is the PV of $1,000 to be received each year for 10 years, starting five years from now if the discount rate is 7%?

 1000--------------------------------------------------------1000

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

PV = 

 = 7,023.58

 (1.07)4

 = 5,358.26

Note that you have to do this in two steps with Excel. First find the PV of the annuity as of time 4 (7,023.58). Then find the PV of that lump sum at time zero. Whenever you do a two-step problem like this in Excel, you should always **reference the cell** with the number you want to input – don’t write it down and then type it in or you will have rounding errors.

## Future Value of an Annuity

FVA = 

How did we derive that formula?

FV = PV (1+r)t

Substitute PV of an annuity in for PV above:

FV =  (1+r)t

 = 

 = 

**Example**:

Invest $1,000 each month into my 401(k) plan at work. If we assume that our return will be 6% per year (APR), how much will be in our 401(k) after 35 years?

Note that 6% APR means 0.5% per month (6%/12 = 0.5%) and 35 years means 420 months.

FVA = $1,000  = $1,424,710.30

**Combining PV and FV in one Problem**

How much do you need to save each year for 10 years, so that you can pay for 4 years of college at $30,000 per year with an interest (investment) rate of 6%?

 30 30 30 30

\_\_\_\_\_C\_\_\_C\_\_\_C\_\_\_C\_\_C\_\_\_C\_\_\_C\_\_\_C\_\_C\_\_\_C\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

0 1 2 3 4 5 6 7 8 9 10 11 12 13

PVAD = $(1.06)

 = $110,190.36 at year 10

$110,190.36 = C 

C = $ 8,359.92

**Present Value of a Growing Perpetuity**

Suppose you are getting a dividend payment of $100 per year and you expect the company’s dividends to grow at the rate of 3% per yer forever. What is the present value of this growing perpetuity if the appropriate interest rate is 12%?

PVGP = \_\_C1\_\_

 r-g

PVGP = 100

 .12-.03

PVGP = 1111.11

Note that if g = 0, we have our standard perpetuity formula.

## Present Value of a Growing Annuity

Suppose you feel that the company is growing its dividends too fast and will likely dissolve in 10 years. What is the present value of this growing annuity?

PVGA = 630.33

There is no pre-programmed Present Value of a Growing Annuity formula in Excel, but that doesn’t prevent you from writing one yourself!

Another way to do this in Excel is to type the first cash flow into cell A1. In cell B1, type “=A1\*1.03”. Now drag the contents of that cell out to cell J1 so that the value in each cell is 3 percent more than the value in the prior cell. Now that you have all the cash flows from $100 to $130.48, find the NPV of those cash flows at a discount rate of 12%. It should give you $630.33.